On a family of subgroups of the Teichmuller modular group of genus two obtained from the Jones representation

AUTHOR(S):

MORISHITA, Masanori

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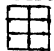
Masanori MORISHITA
Department of Mathematics, Faculty of Science
Kanazawa University

Introduction

The purpose of the present paper is to give a family of "non-Torelli" subgroups of the Teichmüller modular group of genus 2 by confirming a conjecture, posed by Takayuki Oda, on the image of the Jones representation.

In [J], Jones attached to a Young diagram a Hecke algebra representation of the braid group B_n on n strings. As was shown in [ibid,10], the Jones representation of B_6 corresponding to the rectangular Young diagram  factors through the Teichmüller modular group Γ of genus 2, namely, the mapping class group of a closed orientable surface of genus 2, and we thus get the representation $\pi : \Gamma \longrightarrow GL_5(\mathbf{Z}[x, x^{-1}])$ which is explicitly given ([ibid, p362). Now, for a certain natural number n, specializing x to $exp(2\pi\sqrt{-1}/n)$, we get a representation $\pi_n : \Gamma \longrightarrow GL_5(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in the n-th cyclotomic field K. Let F be the maximal real subfield of K and take a non-zero ideal I of $\mathcal{O}_F$, the ring of integers of F. The reduction of π_n modulo $I_K = I\mathcal{O}_K$ gives a representation $\pi_{n,I} : \Gamma \longrightarrow GL_5(\mathcal{O}_K/I_K)$. Then, Oda conjectured that the image of $\pi_{n,I}$ is a certain unitary group if I is prime to an ideal of $\mathcal{O}_F$ containing (n). (For the precise formulation, see Section 2).

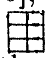
The main result of this paper is to confirm Oda's conjecture when I is a product of prime ideals of $\mathcal{O}_F$ which are inert in K/F. The proof consists of two steps. We first show that $\pi_{n,\wp}$ is irreducible under certain conditions on n and a prime $\wp$, and then investigate the list of all irreducible subgroups of $PSL_5(\mathcal{O}_K/\wp_K)$ due to Martino and Wagoner [M-W]. For the case of a product of inert primes, we apply a criterion of Weisfeiler on the approximation of a Zariski-dense subgroup in a semisimple group over a finite ring [W]. This proof is similar to that of Oda and Terasoma ([O-T]) for the similar problem on the Burau representations, where they use the induction after working with 2×2 matrices (see also [Be]). Our case is more complicated, for we work with 5×5 matrices and so the finite group theory is more involved.

We also check that the kernel of $\pi_{n,I}$ does not contain the Torelli group using its explicit generator given by Birmann [B1].

Since the Teichmüller modular group is the fundamental group of the moduli space $\mathcal{M}$ of compact Riemann surfaces of genus 2, our result gives a tower of 3-folds, namely, finite Galois coverings of $\mathcal{M}$ with the Galois groups of finite unitary groups.

Notation. For an associative ring R with identity , $M_n(R)$ denotes the total matrix algebra over R of degree n, and $GL_n(R)$ denotes the groups of invertible elements of $M_n(R)$. We write $R^\times$ for $GL_1(R)$. For $A \in M_n(R)$, ${}^tA$, $tr(A)$, and $det(A)$ stand for the transpose, trace, and determinant of A, respectively. We write 0_n and 1_n for the zero and identity matrix in $M_n(R)$, respectively, and e_{ij} for the matrix unit and $diag(\cdot)$ for the diagonal matrix.

1. The Jones representation of the Teichmüller modular group of genus 2 and its unitarity

In [J], Jones attached to each Young diagram with n tiles a Hecke algebra representation of the braid group B_n on n strings. As was shown in [ibid, Section 10], the representation of B_6 corresponding to the rectangular Young diagram  factors through the Teichmüller modular group Γ of genus 2, namely, the mapping class group of a closed orientable surface of genus 2. It is known that Γ admits the following presentation ([Bi2], Theorem 4.8, p 183-4).

generators: $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$.

defining relation:

$$\begin{cases} \theta_i\theta_{i+1}\theta_i = \theta_{i+1}\theta_i\theta_{i+1} \ \ (1 \le i \le 4), \\ \theta_i\theta_j = \theta_j\theta_i \ \ (|i-j| \ge 2, 1 \le i,j \le 5), \\ (\theta_1\theta_2\theta_3\theta_4\theta_5)^6 = 1, \\ (\theta_1\theta_2\theta_3\theta_4\theta_5^2\theta_4\theta_3\theta_2\theta_1)^2 = 1, \\ \theta_1\theta_2\theta_3\theta_4\theta_5^2\theta_4\theta_3\theta_2\theta_1 \text{ commutes with } \theta_i \ \ (1 \le i \le 5). \end{cases}$$

The Jones representation of Γ mentioned above is given explicitly on generators as follows ([J], p362).

$$\pi : \Gamma \longrightarrow GL_5(\mathbf{Z}[x, x^{-1}]), \ \ x = t^{1/5};$$

$$\pi(\theta_1) = x^{-2}\begin{pmatrix} -1 & 0 & 0 & 0 & t \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & t \end{pmatrix}, \ \pi(\theta_2) = x^{-2}\begin{pmatrix} t & 0 & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & t & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\pi(\theta_3) = x^{-2}\begin{pmatrix} -1 & 0 & 0 & t & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & t & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}, \ \pi(\theta_4) = x^{-2}\begin{pmatrix} t & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & t \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & t \end{pmatrix},$$

$$\pi(\theta_5) = x^{-2}\begin{pmatrix} -1 & t & 0 & 0 & 0 \\ 0 & t & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}.$$

We see that $\det\pi(\theta_i) = -1$, $1 \le i \le 5$.

Let $A = A(x) \in M_n(\mathbf{Z}[x, x^{-1}]), x = t^{1/5}$. We write A^* for ${}^tA(x^{-1})$ and call A x-hermitian if $A = A^*$. For a t-hermitian matrix A, we define the unitary group with respect to A by

$$U_n(A) := \{g \in GL_n(\mathbf{Z}[x, x^{-1}]) | g^*Ag = A\}.$$

Lemma 1.1. *Let π be the representation given in Section* 1. *Then, there is a t-hermitian matrix $H \in M_5(\mathbf{Z}[x, x^{-1}])$ so that the image of π is contained in $U_5(H)$.*

Proof. By the straightforward computation, the following x-hermitian matrix satiafies the desired property.

$$\begin{pmatrix} (1+t)(1+t^{-1}) & -(1+t) & 2 & -(1+t) & -(1+t) \\ -(1+t^{-1}) & 1+t+t^{-1} & -(1+t^{-1}) & 1 & 1 \\ 2 & -(1+t) & (1+t)(1+t^{-1}) & -(1+t) & -(1+t) \\ -(1+t^{-1}) & 1 & -(1+t^{-1}) & 1+t+t^{-1} & 1 \\ -(1+t^{-1}) & 1 & -(1+t^{-1}) & 1 & 1+t+t^{-1} \end{pmatrix}.$$

If H' is such a matrix, then $H'H^{-1}$ commutes with $\pi(\theta_i), 1 \leq i \leq 5$. By the computation, we check that $H'H^{-1} \in \mathbf{Q}(x)^{\times} 1_5$.

We write $h = h_t$ for the matrix in the proof. We see that $\det(h_t) = (t + t^{-1})^4(1 + t + t^{-1})$.

2. The reduction of the specialized Jones representation at root of unity and the conjecture of Oda

Let n be a natural number. We assume that n is bigger than 2 and prime to 10. Let $\eta = exp(2\pi\sqrt{-1}/n)$ and $\zeta = \eta^5$. Set $K = \mathbf{Q}(\zeta), \mathcal{O}_K = \mathbf{Z}[\zeta], F = \mathbf{Q}(\zeta + \zeta^{-1})$ and $\mathcal{O}_F = \mathbf{Z}[\zeta + \zeta^{-1}]$.

By specializing $t \to \zeta, x = t^{1/5} \to \eta$ in the representation π, we get a representation

$$\pi_n : \Gamma \longrightarrow GL_5(\mathcal{O}_K).$$

Take a non-zero ideal I of $\mathcal{O}_F$ which is prime to n, and set $I_K = I\mathcal{O}_K$. The reduction of π_ζ modulo I_K defines the representation

$$\pi_{n,I} : \Gamma \longrightarrow GL_5(\mathcal{O}_K/I_K).$$

Then, $\pi_{n,I}$ certainly inherits the unitarity from π.

Lemma 2.1. *The image of $\pi_{n,I}$ is contained in*

$$U_5(\mathcal{O}_K/I_K; h_{n,I}) := \{g \in GL_5(\mathcal{O}_K/I_K) \mid g^* h_I g = h_I\},$$

where $h_{n,I} := h_\zeta \bmod I_K$ and $g^* = {}^t g^\tau$, τ is the involution induced from the generator of $\mathrm{Gal}(K/F)$.

Proof. Immediate from Lemma 1.1. □

To formulate the conjecture, we twist π_I a little bit. Let $\chi : \Gamma \to \mathcal{O}_K^\times$ be the character defined by $\chi(\theta_i) = -1$, and set $\chi_I := \chi \bmod I_K$. We then consider $\rho_I := \pi_{n,I} \otimes \chi_I$. Since det $(\pi_\zeta(\theta_i)) = -1$, by Lemma 2.1, we have the inclusion

$$\rho_I(\Gamma) \subset SU_5(\mathcal{O}_K/I_K; h_{n,I}) := \{g \in U_5(\mathcal{O}_K/I_K; h_{n,I}) \mid \det(g) = 1\}.$$

Then, the conjecture posed by Oda is formulated as follows.

Conjecture 2.2. *There is a non-zero ideal $\mathcal{C}$ of $\mathcal{O}_F$ containing (n) so that the image of $\rho_{n,I}$ coincides with $SU_5(h_{n,I})$ if I is prime to $\mathcal{C}$.*

3. Non-split prime case

In this section, we verify Conjecture 2.2, when I is a maximal ideal $\wp$ of $\mathcal{O}_F$ which is inert in K/F. Set $\mathbf{F}_\wp = \mathcal{O}_F/\wp, \mathbf{F} = \mathbf{F}_{\wp K} = \mathcal{O}_K/\wp_K$ for simplicity. We simply write $\pi_\wp$ and $\rho_\wp$ for $\pi_{n,\wp}$ and $\rho_{n,\wp}$, respectively, also $h_\wp$ for $h_{n,\wp}$.

First, the following lemma shows each $\pi_\wp(\theta_i)$ is a quasi-reflection.

Lemma 3.1. *Assume that $\wp$ is prime to $1+\zeta$. Let $V = \mathbf{F}^{\oplus 5}$ be the representation space of $\pi_\wp$. For each $1 \le i \le 5$, there are subspaces X_i and Y_i of V such that*

$$V = X_i \oplus Y_i, \qquad dim X_i = 3,\ dim Y_i = 2,$$
$$\pi_\wp(\theta_i)|_{X_i} = -\eta^{-2} id_{X_i}, \qquad \pi_\wp(\theta_i)|_{Y_i} = \eta^3 id_{Y_i},$$

where η denotes a primitive n-th root of 1 in $\mathbf{F}$ *by abuse of notation.*

Proof. By the direct computation, X_i and Y_i are given as follows:

$$\begin{array}{ll}
X_1 = \{{}^t(x_1, x_2, 0, x_4, 0)\}, & Y_1 = \{{}^t(y_1, y_2, (1+\zeta)y_2, y_2, (1+\zeta^{-1})y_1)\} \\
X_2 = \{{}^t(0, 0, x_3, x_4, x_5)\}, & Y_2 = \{{}^t((1+\zeta)y_1, (1+\zeta^{-1})y_2, y_2, y_1, y_1)\} \\
X_3 = \{{}^t(x_1, x_2, 0, 0, x_5)\}, & Y_3 = \{{}^t(y_1, y_2, (1+\zeta)y_2, (1+\zeta^{-1})y_1, y_2)\} \\
X_4 = \{{}^t(0, x_2, x_3, x_4, 0)\}, & Y_4 = \{{}^t((1+\zeta)y_1, y_1, y_2, y_1, (1+\zeta^{-1})y_2)\} \\
X_5 = \{{}^t(x_1, 0, 0, x_4, x_5)\}, & Y_5 = \{{}^t(y_1, (1+\zeta^{-1})y_1, (1+\zeta)y_2, y_2, y_2)\},
\end{array}$$

where x_i's and y_i's run over $\mathbf{F}$ and $\zeta = \eta^5$. $\square$

Lemma 3.2. *Assume that $\wp$ is prime to $(1+\zeta)(\zeta+\zeta^{-1})(1+\zeta+\zeta^{-1})$. Then, the representation $\pi_\wp$ is irreducible.*

Proof. Suppose that V has $\pi_\wp(\Gamma)$-invariant subspace $W \neq 0, V$. First, assume $\dim(W) = 1$. Let w be a base of W and write $w = x+y, x \in X_1, y \in Y_1$. If $\pi_\wp(\theta_1)w = \alpha w, \alpha \in \mathbf{F}^\times$, by Lemma 4.1, we have $(\alpha+\eta^2)x + (\alpha-\eta^3)y = 0$, from which we see that $w \in X_1$ or $w \in Y_1$. Let $w = {}^t(x_1, x_2, 0, x_4, 0) \in X_1$. Then, $\pi_\wp(\theta_2)w = \eta^{-2}\, {}^t(\zeta x_1, \zeta x_2, \zeta x_2, x_1 - x_4, x_1)$ should be in X_1 and so we get $w = 0$. This is a contradiction. Similarly, w can not be in Y_1. Hence, $\dim(W) > 1$. Note that the hermitian form $h_{n,\wp}$ is non-degenerate by our assumption. So, we may assume $\dim(W) = 2$, since the orthogonal complement of W with respect to $h_{n,\wp}$ is $\pi_\wp(\Gamma)$-invariant. For this case, consider the exterior square representation $\bigwedge^2 \pi_\wp : \Gamma \longrightarrow GL(\bigwedge^2 V)$. Then, $\bigwedge^2 W$ is an invariant subspace of $\bigwedge^2 V$ and $\dim(\bigwedge^2 W) = 1$, and the similar argument to the above can be applied. Fix a basis of X_1; $v_1 = {}^t(1,0,0,0,0), v_2 = {}^t(0,1,0,0,0), v_3 = {}^t(0,0,0,1,0)$ and a basis of Y_1; $v_4 = {}^t(1,0,0,0,1+\zeta^{-1}), v_5 = {}^t(0,1,1+\zeta,1,0)$ and set $V_1 = \mathbf{F}v_1 \wedge v_2 + \mathbf{F}v_2 \wedge v_3 + \mathbf{F}v_1 \wedge v_3$, $V_2 = \mathbf{F}v_4 \wedge v_5$, and $V_3 = \mathbf{F}v_1 \wedge v_4 + \mathbf{F}v_1 \wedge v_5 + \mathbf{F}v_2 \wedge v_4 + \mathbf{F}v_2 \wedge v_5 + \mathbf{F}v_3 \wedge v_4 + \mathbf{F}v_3 \wedge v_5$. Then, we get the decomposition $\bigwedge^2 V = V_1 \oplus V_2 \oplus V_3$, and by Lemma 4.1, $\pi_\wp(\theta_1)$ acts on V_1, V_2, V_3 by the scalar multiples $\eta^{-4}, \eta^6, -\eta$, respectively, from which we see that $\bigwedge^2 W$ sits in one of V_i's. Suppose $W = \mathbf{F}w \subset V_1$. Then, $\bigwedge^2 \pi(\theta_j)w$, $2 \leq j \leq 5$, should be in V_1. Using the above base of V_1 and the assumption on $\wp$, just write down these and we get $w = 0$. Similarly, W can't be in V_2, V_3. We conclude $\pi_\wp$ is irreducible. $\square$

Now, we shall determine the image of $\rho_\wp$ and there is a list of irreducible subgroups of $PSL_5(\mathbf{F})$ due to Martino and Wagoner [M-W]. Here, we assume further that $\wp$ is prime to 2. By abuse of notation we write $\rho_\wp$ for the associated projective representation and set $G = \rho_\wp(\Gamma)$, which is an irreducible subgroup of $PSL_5(\mathbf{F})$ by Lemma 3.2.

First, we have the following

Lemma 3.3. *The group G can not be realized over $\mathbf{F}_{p^a}, a < 2f$, where p^{2f} is the cardinality of $\mathbf{F}$*

Proof. Suppose that G is a subgroup of $PSL_5(\mathbf{F}_{p^a}), a < 2f$. Then, the characteristic polynomial $(X - \eta^{-2})(X + \eta^3)$ of $\rho_\wp(\theta_1)$ is invariant under the action of the Galois group $\mathrm{Gal}(\mathbf{F}_{p^{2f}}/\mathbf{F}_{p^a}) = <\sigma>$, where $\sigma =$ Frobenius automorphism, and so $\eta^\sigma = \eta^{p^a}$, by $(\eta^{-2})^\sigma = \eta^{-2}$. Since $(n, 10) = 1$, $p^a \equiv 1$ mod n. This contradicts to the minimality of $2f$ so that $p^{2f} \equiv 1$ mod n. □

By Lemma 3.2, the following groups in the list of Martino-Wagoner can not be G: (1.3)-(a), (1.5), (1.7), (1.10)-(a), (1.12), (1.13), (1.14)-(a), (1.15), (1.16), where the numbers are those in [M-W].

Next, since the image of $\rho_\wp$ is contained in $SU_5(\mathcal{O}_K/\wp_K; h_\wp) \simeq SU_5(\mathbf{F})$, G can not be $PSL_5(\mathbf{F}), PSO_5(\mathbf{F})$ and $P\Omega_5(\mathbf{F})$, by comparing the orders. So, the groups (1.4), (1.8), (1.9) and (1.10)-(b) in [M-W] are excluded.

The following useful lemma was suggested by Eiichi Bannai.

Lemma 3.4. *The subgroup of G generated by $\rho_\wp(\theta_1)$ and $\rho_\wp(\theta_3)$ is isomorphic to $\mathbf{Z}/2n\mathbf{Z} \times \mathbf{Z}/2n\mathbf{Z}$.*

Proof. By Lemma 3.1, the order of $\rho_\wp(\theta_i)$ is $2n$. We easily check $< \rho_\wp(\theta_1) > \cap < \rho_\wp(\theta_3) >= id$. □

The group (1.2) in [M-W] is a subgroup of the group which is an extension of a cyclic subgroup by $\mathbf{Z}/5\mathbf{Z}$. So, by Lemma 3.4, G can not be this group. Next, (1.11) is $PSL_2(\mathbf{F})$ or $PGL_2(\mathbf{F})$. We have a list of subgroups of $PSL_2(\mathbf{F})$ due to Dickson, [H], p213, Satz 8.27. Looking at this, by Lemma 3.3, G can not be a subgroup of $PSL_2(\mathbf{F})$. Since $PGL_2(\mathbf{F})$ is an extension of $PSL_2(\mathbf{F})$ by a cyclic subgroup of order 2, G can't be in $PGL_2(\mathbf{F})$. The similar argument can be applied to the groups (1.3)-(b),(c).

Finally, the group (1.1) can be excluded as follows (E. Bannai). The group (1.1) is an irreducible subgroup of A, where A is a global stabilizer in $PSL_5(\mathbf{F})$ of a simplex. Note that A is a monomial group and has a normal subgroup N so that $A/N \simeq S_5$= the symmetric group on 5 letters. Assume that G is an irreducible subgroup of A. Then, $\bar{G} = G/(G \cap N)$ is a subgroup of S_5 and then $\bar{G}$ can be one of S_5, A_5, Frobenius group of order 20, dihedral group of order 10, or cyclic group of order 5. The images of $\rho_\wp(\theta_i)$ in $\bar{G}$ satisfy the relation induced from that of the mapping class group, from which we can conclude $\bar{G}$ is cyclic of order 5. This is a contradiction by the assumption $(n, 10) = 1$.

Summing up the above, we have

Theorem 3.5. *Assume that n is prime to* 10 , *bigger than* 2 *and that a prime ideal* $\wp$ *of* $\mathcal{O}_F$ *does not divide* $2(1+\zeta)(\zeta+\zeta^{-1})(1+\zeta+\zeta^{-1})$ *and is inert in* K/F. *Then, the image of* $\rho_\wp$ *coincides with* $SU_5(\mathcal{O}_K/\wp_K; h_\wp)$.

4. The case of a product of non-split primes

In this section, we extend Theorem 3.5 to the case where I is a product of non-split primes. For this, we apply a criterion of Weisfeiler on the approximation of a Zariski-dense subgroup in a semisimple group over a finite ring to our situation. In the following, we simply call (i) $\sim$ (iv) for Weisfeiler's assumptions (i) $\sim$ (iv) in (7.1) of [W].

Let I be a product of different prime ideals $\wp_i$ of $\mathcal{O}_F$, $I = \prod_{i=1}^r \wp_i^{e_i}$, where each $\wp_i$ is inert in K/F and prime to $6(1+\zeta)(\zeta+\zeta^{-1})(1+\zeta+\zeta^{-1})$. Set $A = \mathcal{O}_F/I$ and $B = \mathcal{O}_K/I_K, I_K = I\mathcal{O}_K$. Write $\mathbf{F}_{q_i} = \mathcal{O}_F/\wp_i, q_i = N\wp_i$, for simplicity. The radical of A is $R = \prod_{i=1}^r \wp_i$.

Let G_h and G be the special unitary group schemes over A with respect to the hermitian forms $h_I = h_\zeta \bmod I_K$ and $1_5 \in M_5(B)$ on the free B-module $M = B^{\oplus 5}$, respectively.

Our task is to show $G_h(A) = \rho_I(\Gamma)$. Fixing an isometry $\phi : (M; h_I) \simeq (M; 1_5)$ of hermitian modules, it is reduced to show $G(A) = \Gamma'$, where $\Gamma' = \phi\rho_I(\Gamma)\phi^{-1}$.

Let T_1 be the norm 1 torus attached to the quadratic extension B/A;

$T_1 := \mathrm{Ker}(R_{B/A}(\mathbf{G_m}_{,B}) \xrightarrow{N} \mathbf{G_m}_{,A})$, where $\mathbf{G_m}$ is the split multiplicative group scheme of dimension 1 and $R_{B/A}$ is the Weil restriction of the scaler, and N is the norm map attached to B/A.

A maximal A-torus of G is given by $T := \{t = diag(t_1, t_2, t_3, t_4, t_5)|\ t_i \in T_1, \prod_{i=1}^{5} t_i = 1\}$. Fix an isomorphism $T_1 \simeq \mathbf{G_m}$ over B and define the character χ_i of T by $\chi_i(t) := t_i$, $1 \le i \le 4$. Then, the character module $X^*(T)$ of T is generated by $\chi_i, 1 \le i \le 4$. Suppose that $\chi|_{T(\mathbf{F}_{q_i})} = \chi'|_{T(\mathbf{F}_{q_i})}$ for $\chi, \chi' \in X^*(T)$. Then, writing χ and χ' as products of powers of χ_i's, we easily see that $\chi = \chi'$. So, the assumption (i) is just $q_i \ge 10, 1 \le i \le r$. The assumption (ii) is satisfied for our G and (iii) is a consequence of Theorem 3.5.

Finally, let $Ad : G(A) \to GL(L(A))$ be the adjoint representation, where L is the Lie algebra of G and given by $L(A) = \{X \in M_5(B)|\ tr(X) = 0,\ {}^tX^\sigma + X = 0\}$. Write $B = A + A\beta$, $\beta^2 \in A$, and take $\beta(e_{11} - e_{55}), \cdots, \beta(e_{44} - e_{55}), e_{ij} - e_{ji}, \cdots, \beta(e_{ij} + e_{ji})$, $(i < j)$ as a basis of $L(A)$. Using this basis, a straightforward calculation shows that $tr(Ad(g)) = N_{B/A}(tr(g)) - 1$ for $g \in G(A)$, where $N_{B/A}$ is the Norm map attached to B/A and $N_{B/A}(tr(\rho_I(\theta_1))) = 13 - 6(\zeta + \zeta^{-1})$. From this, we get $\mathbf{Z}[tr Ad(\Gamma') \mathrm{mod}\ R^2] = A/R^2$ which certifies the assumption (iv).

Summing up the above, we have

Main Theorem 4.1. *Let I be a product of prime ideals $\wp_i$ of $\mathcal{O}_F$. Assume that each $\wp_i$ is inert in K/F and prime to $6(1+\zeta)(\zeta+\zeta^{-1})(1+\zeta+\zeta^{-1})$ and $N\wp_i \ge 10$. Then, the image of ρ_I coincides with $SU_5(\mathcal{O}_K/I_K, h_I)$.*

5. Comparison with the Torelli group and coverings of the moduli space of compact Riemann surfaces of genus 2

Let $Sp_2(\mathbf{Z})$ be the Siegel modular group of degree 4, namely, the group consisting of all $S \in GL_n(\mathbf{Z})$ such satisfing

$$SJ\,{}^tS = J, \quad J = \begin{pmatrix} 0_2 & 1_2 \\ -1_2 & 0_2 \end{pmatrix}.$$

Let $\theta : \Gamma \to Sp_2(\mathbf{Z})$ be a canonical homomorphism induced by the abelianization map of Γ and the Nielsen isomorphism. We call the kernel of θ the

Torelli group of genus 2 and write $\Gamma(N)$ for $\theta^{-1}(Sp_2(\mathbf{Z};N))$, where $Sp_2(\mathbf{Z};N)$ is the principal congruence subgroup of $Sp_2(\mathbf{Z})$ modulo a natural number N. The following result of Birmann allows us to compare our groups $\Gamma_{n,I}$ with the Torelli group and $\Gamma(N)$.

Lemma 5.1.([Bi1], Theorem 2) *The Torelli group of genus 2 is generated by the normal closure of* $(\theta_1\theta_2\theta_1)^4$.

Proposition 5.2. *Under the same assumption in Theorem 4.1, the group* $\Gamma_{n,I}$ *does not contain the Torelli group, hence any* $\Gamma(N)$.

Proof. It is straightforward to check that $\rho_{n,I}((\theta_1\theta_2\theta_1)^4) \neq 1$. □

The geometrical interpretation of the above result is as follows.

Let $\mathcal{T}$ be the Teichmüller space of genus 2 and $\mathcal{M} = \mathcal{T}/\Gamma$ be the moduli space of compact Riemann surfaces of genus 2. Let $\mathcal{S}$ be the Siegel upper half space of degree 4 and $\mathcal{A} = \mathcal{S}/Sp_2(\mathbf{Z})$ be the moduli space of principally polarized abelian varieties. The period map $\mathcal{T} \to \mathcal{S}$ is compatible with the actions of $\Gamma, Sp_2(\mathbf{Z})$ and θ, and thus we obtain the Torelli map $\mathcal{M} \longrightarrow \mathcal{A}$.

The Galois covering $\mathcal{A}_N = \mathcal{S}/Sp_2(\mathbf{Z};N)$ over $\mathcal{A}$ with the Galois group $Sp_2(\mathbf{Z}/N\mathbf{Z})$ is the moduli space of principally polarized abelian varieties with level N-structure. Then, Corollary 5.2 tells us that the spaces $\mathcal{T}/\Gamma_{n,I}$ give a family of Galois coverings over $\mathcal{M}$ with the Galois groups $SU_5(\mathcal{O}_K/I_K)$, which can not be obtained by the pull-back of any $\mathcal{A}_N$ via the Torelli map.

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